GRADUALLY VARIED FLOWS

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Gradually varied flows are open-channel flows whose water surface profile changes gradually. This means that, unlike rapidly varied flows, the hydrostatic pressure distribution is valid along the length of the flow. Both energy approach and momentum approach can be used for derivation, resulting in energy slope and friction slope, respectively. The two derivations provide different insights for gradually varied flows and understanding the difference between the two slopes is important. Features on some gradually varied flows are given and discussed

1. Introduction

Gradually varied flow is the steady flow whose water surface changes gradually along the length of the channel. This means two assumptions are involved in the definition: (1) the flow is steady - the hydraulic characteristics remain constant for the time interval and (2) the streamlines are practically parallel - the hydrostatic pressure distribution prevails over the channel section.

2. Energy Approach

Consider a control volume of water column in the figure below. The total head (H) is

$$H = z + d\cos\theta + \alpha\frac{V^2}{2g} \tag{1}$$

where z = elevation of a channel bottom from a certain datum, d = water depth normal to the water surface, α = energy correction factor, and V = mean velocity. Assuming that α = 1 and the slope is very mild, then $\cos\theta \approx 1$ and $d \approx y$ (water depth vertically from the bed). Differentiating Eq.(1) with respect to x yields

$$\frac{dH}{dx} = -S_e = -S_0 + \frac{dE_s}{dx} \tag{2}$$

where S_e = the slope of the energy (grade) line. It should be noted in the above equation that the change of the total head (dH) is always negative (loss) in the flow direction. Yen and Wenzel (1970) showed that the slope of the energy line (S_e) is not the same as the friction slope (S_f) or the energy dissipation gradient, in general. Rewriting Eq.(2), we have

$$\frac{dE_s}{dx} = S_0 - S_e \tag{3}$$

which can be written as

$$\frac{dy}{dx} = \frac{S_0 - S_e}{1 - Fr^2} \tag{4}$$

since

$$\frac{dE_s}{dx} = \frac{dE_s}{dy}\frac{dy}{dx} \tag{5}$$

$$\frac{dE_s}{dx} = 1 - Fr^2 \tag{6}$$

Eq.(4) is the governing equation for gradually varied flow, also referred to as "backwater equation." It describes the variation of the flow depth in a channel of arbitrary shape. Gradually varied flow should imply that the water depth does not change significantly in order to satisfy $dA/dy = T$ in deriving $dE_s / dx = 1 - Fr^2$.

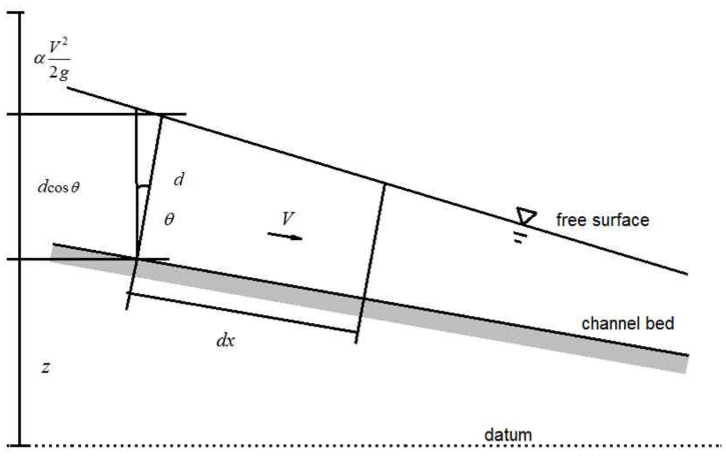

Figure 1. Flow in a rectangular prismatic open-channel

Yen and Wenzel (1970) presented a rigorous derivation of the governing equation for the spatially varied flow using the energy approach. For the case without the lateral inflow, the equation is given by

$$\frac{dy}{dx}=\frac{S_0-S_E}{1-\dfrac{\alpha V^2}{gD}}$$

where S_E is the energy dissipation gradient defined by

$$S_E=-\frac{1}{\gamma AV}\frac{dE}{dx}$$

which accounts for the dissipation of the mechanical energy of the mean motion within the fluid volume due to the fluid viscosity and turbulence. Thus, it can be said that Eq.(4) is not obtained from the energy approach in a rigorous sense. However, herein, Eq.(4) is taken to be the outcome of the energy approach since the total head by Eq.(1) is related to the energy of the open-channel flow.

3. Momentum Approach

The derivation of the gradually varied flow equation from the force balance provides a better insight of the weakness of the backwater equation in the applications. For example, hydrostatic pressure distribution, which is extremely critical in practice, cannot be seen in the derivation from the energy concept.

For the fluid element between sections 1 and 2, the change of momentum flux in the *x*-direction is

$$\sum F = \beta\rho Q\left(V_2 - V_1\right) \tag{7}$$

where β is the momentum correction factor. The RHS of Eq.(7) is given by

$$\sum F = \gamma S_0 A dx - \tau_0 P dx - \int_A p dA \Big|_1^2 \tag{8}$$

If one divides Eq.(8) by *dx* and takes the limit as *dx* approaches zero, then Eq.(7) becomes

$$\gamma S_0 A - \tau_0 P - \frac{d}{dx}\int p dA = \beta\rho Q \frac{dV}{dx} \tag{9}$$

where the third term on the LHS is given by

$$\int p dA = K\gamma A y \cos\theta \tag{10}$$

where K is the pressure coefficient. Note that even if the pressure distribution is hydrostatic, the value of K varies with the cross sectional shape and is $K = 0.5$ only for a 2D flow. By using such relationships as

$$V\frac{dA}{dx} + A\frac{dV}{dx} = 0 \qquad \text{from continuity} \tag{11}$$

and

$$\frac{dA}{dx} = \frac{A}{D}\frac{dy}{dx} \tag{12}$$

Eq.(9) can be rewritten as

$$\frac{dy}{dx}\left[K\cos\theta(1+\frac{y}{D}) - \frac{\beta V^2}{gD}\right] = S_0 - \frac{\tau_0}{\gamma R_h} - y\frac{d(K\cos\theta)}{dx} \tag{13}$$

or

$$\frac{dy}{dx} = \frac{S_0 - \tau_0 / \gamma R_h}{\cos\theta\lambda - \dfrac{\beta V^2}{gD}} \tag{14}$$

where

$$\lambda = K\left(1 + \frac{y}{D}\right) + y\frac{dK}{dy} \tag{15}$$

In deriving above equation, it is assumed that β does not vary with x. If the pressure distribution is hydrostatic, then λ is unity (Yen and Wenzel, 1970). Thus, finally, we have

$$\frac{dy}{dx} = \frac{S_0 - S_f}{\cos\theta\lambda - \dfrac{\beta V^2}{gD}} \tag{16}$$

where

$$S_f = \frac{\tau_0}{\gamma R_h} \tag{17}$$

(Q) Prove that $\lambda = 1$ for hydrostatic pressure distribution.

4. Energy Slope and Friction Slope

In solving GVF problems, it is necessary to evaluate S_f if the momentum equation is used and S_e if the energy equation is used. The friction slope given by Eq.(17) accounts for the resistance due to external boundary shear stress in the x-direction. This value, however, is usually not available and it should be estimated by other means. Common practice is to approximate S_f by using the resistance relationship such as

$$S = \frac{n^2 V^2}{C_m^2 R_h^{4/3}}$$

$$S = \frac{V^2}{C^2 R_h}$$

$$S = \frac{f V^2}{8 g R_h}$$

This is based on the assumption of steady uniform flow where $S_0 = S_w = S_f$.

Evaluation of energy slope S_e requires detailed measurements of flow depth and mean velocity, which is nearly impossible. Similarly, common practice again approximates S_e by the above resistance relationships. However, the assumption of equating two or more gradients ($S_f = S_e = S_H$) is only an approximation although it is often sufficiently accurate for engineering purposes.

(Q) Why the energy slope is frequently looked upon as the friction slope?

5. Classifications of the Gradually Varied Flow

The gradually varied flow equation becomes

$$\frac{dy}{dx} = S_0 \frac{1-(y_n / y)^{10/3}}{1-(y_c / y)^3} \tag{20}$$

if the Manning formula is used for the friction slope (or total head gradient). Similarly, we have

$$\frac{dy}{dx} = S_0 \frac{1-(y_n / y)^3}{1-(y_c / y)^3} \tag{21}$$

if the Chezy formula is used.

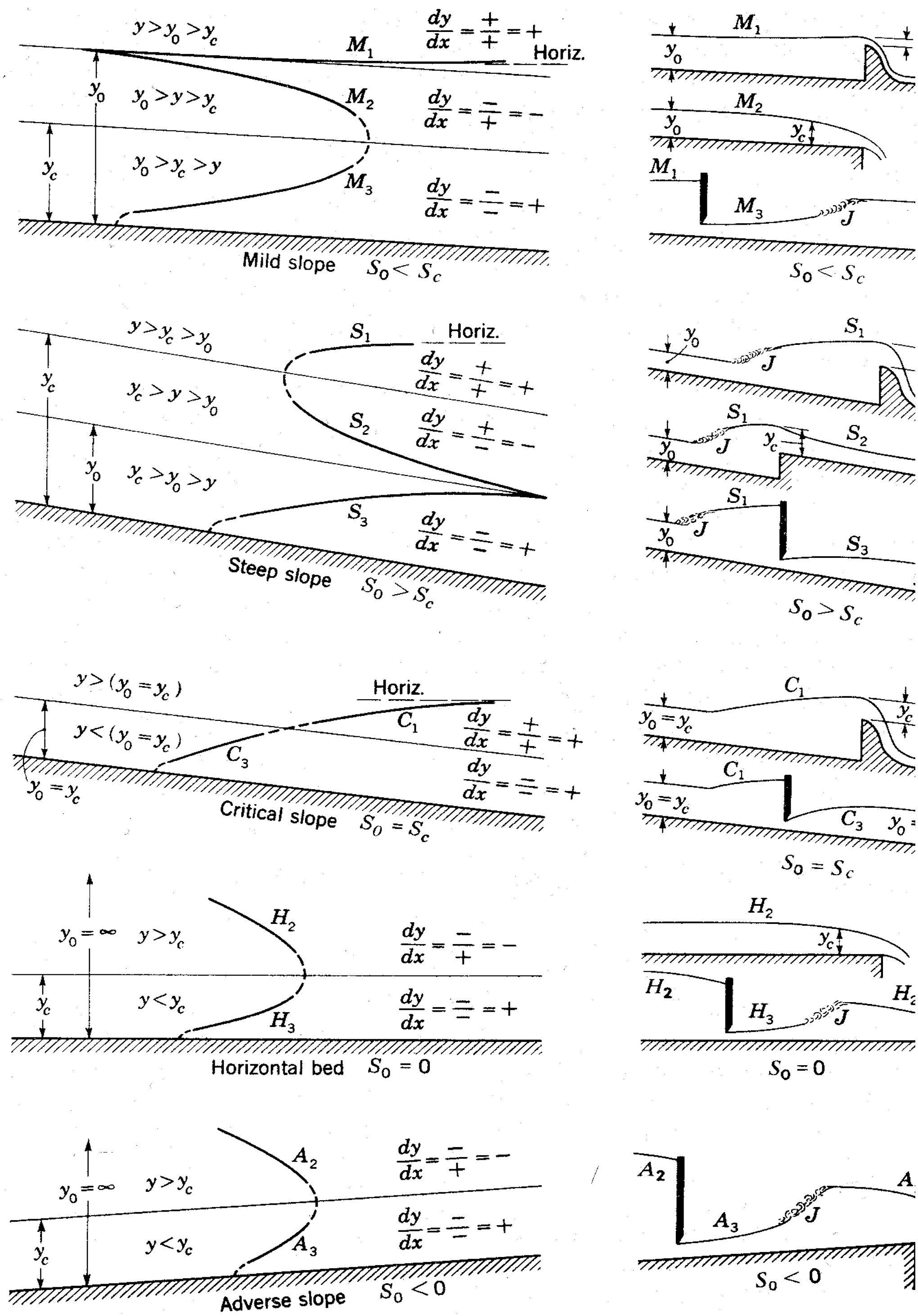

Figure 11.20 Various types of nonuniform flow with flow from left to right.

Figure 2. Various flow profiles of gradually varied flows

6. Some Flow Profiles

6.1 Flow under the Sluice Gate

The flow under the sluice gate on the mild slope is considered. The flow profile upstream of the gate is M1 curve. The flow from the tip of the gate shows a vena contracta because of the generation of the highly-curvilinear flow. Hydrostatic pressure distribution is not valid for this type of flow. Therefore, care must be taken when applying the energy equation. If the gate is located between the bed and y_c, the flow profile is M3 curve. Then, in order to go across the critical depth, the hydraulic jump should take place. If the jump rises to a height between y_c and y_n, then M2 curve is made. Or, if the jump rises to a height beyond y_n, then M1 curve is made. In both cases, the flow depth never approaches y_n far downstream. Therefore, the jump will rise exactly to y_n.

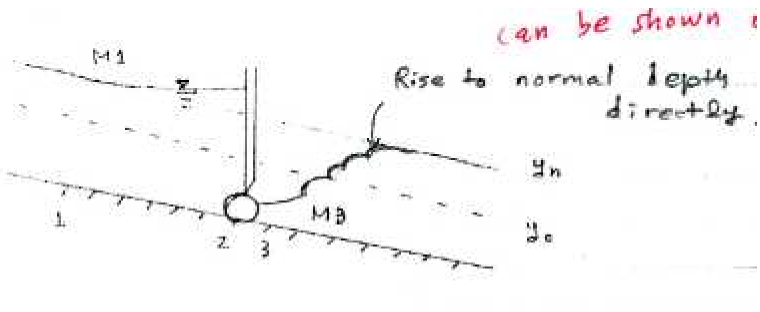

Figure 3. Flow under the sluice gate

6.2 Flow on the changing slope

The flow on the mild slope is considered. If the slope increases slightly, then the normal depth is lowered accordingly, without changing the critical depth. The flow profile upstream of the break in slope shows M2 curve. If the flow depth is located at a height higher than y_n, then M1 curve is made. Or, if the flow depth is located between y_c and y_n, then M2 curve is made. In both cases, the flow depth never approaches y_n far downstream. Therefore, the flow depth at the break in slope drops exactly to y_n.

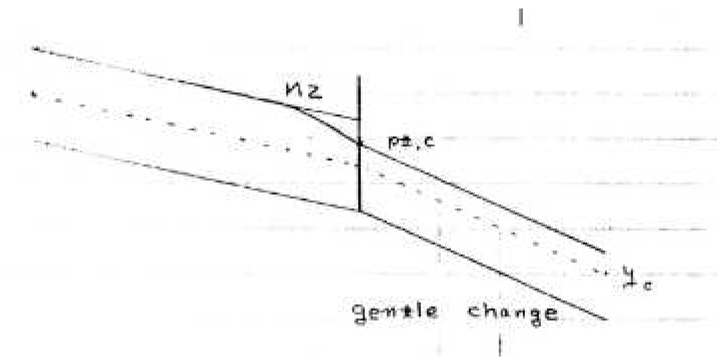

Figure 4. Flow on the increasing slope

If the slope decreases slightly, then the normal depth is lifted accordingly but the critical depth does not change. The flow profile upstream of the break in slope shows M1 curve. If the flow depth is located at a height higher than y_n, then M1 curve is made. Or, if the flow depth is located between y_c and y_n, then M2 curve is made. In both cases, the flow depth never approaches y_n far downstream. Therefore, the flow depth at the break in slope increase exactly to y_n.

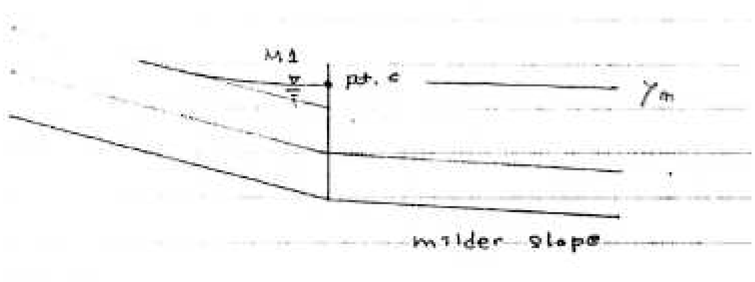

Figure 5. Flow on the decreasing slope

6.3 Flow on the slope changing roughness

The flow in the channel whose roughness changes is considered. If the roughness downstream of the channel decreases significantly, then the normal depth can be located below the critical depth. This is due to that the critical slope is proportional to the square of the roughness ($S_c = n^2 gD / R_h^2$). In this case, the flow profiles upstream and downstream of Section 1 are M curve and S curve, respectively. The flow profile upstream of Section 1 is M2, and the flow depth at Section 1 should be y_c. This is because, if the flow depth is higher than y_c, the flow profile is S1 curve and never approach y_n. If the flow depth is lower than y_c, then the flow profile is S2 curve and never approach y_n downstream of this section.

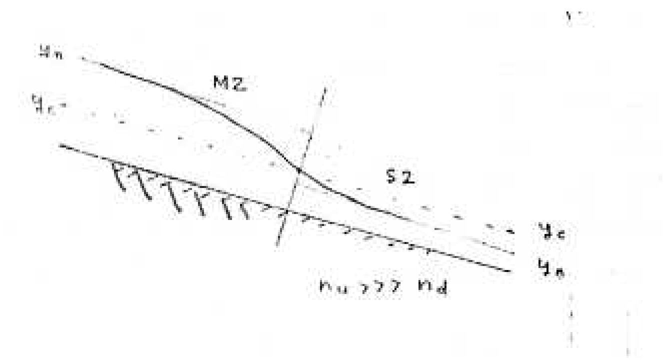

Figure 6. Flow on the slope changing roughness greatly

If the roughness downstream of the channel decreases slightly, then the normal depth is lowered but is higher than y_c. In this case, the both flow profiles upstream and downstream of Section 1 are M curves, and the flow depth at Section 1 is y_n. This is because, if the flow depth is higher than y_n, the flow profile is M1 curve and never approach y_n. If the flow depth is lower than y_n, then the flow profile is M2 curve and never approach y_n downstream of this section.

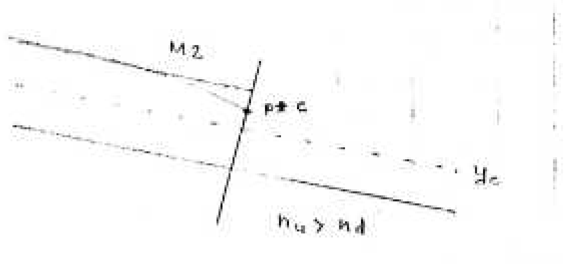

Figure 7. Flow on the slope changing roughness gently

7. Methods of Computation

Computation of gradually varied flow is important in hydraulic engineering practice. Recently, money for the compensation far exceeds money for the construction in building a dam. So accurate computation of the backwater equation under various circumstances is highly desired. Methods for computing the gradually varied flow can be divided into

(1) Direct integration method

- Bresse method
- Chow's method

(2) Step method

- Direct step method
- Standard step method

(3) Numerical method

References

Chow, V.T. (1959). *Open-Channel Hydraulics.* McGraw Hill Book Company, New York, NY.

Yen, B.C., and Wenzel, H.G. (1970). "Dynamic equations for steady spatially varied flow." *Journal of the Hydraulics Division*, ASCE, 96(HY3), 801-814.

Sabersky, R.H., Acosta, A.J., and Hauptmann, E.G. (1971). *Fluid Flow*. Macmillan, New York, NY.

Problems

1. **Gradually Varied Flow Profile** (Julien, p.49)

Consider steady flow in the following impervious rigid boundary channel. The discharge per unit width q is 3.72 m^2/s and the water depth at the downstream dam site is 10 m. Assume that the channel width remains large and constant regardless of flow depth, and f = 0.03. Determine the distribution of the flow depth, mean velocity, and bed shear stress along the 25 km reach of the channel using the numerical method (Newton-Raphson method). Obtain the flow profiles using the following step methods and compare the results:

(a) direct step method

(b) standard step method